# Surface Recovery from Planar Sectional Contours

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**Abstract**

In this paper, we propose a new approach for surface recovery from planar sectional contours. The surface is reconstructed based on the so-called "Equal Importance Criterion," which suggests that every point in the region contributes equally to the reconstruction process. The problem is then formulated in terms of a partial differeXntial equation, and the solution is efficiently calculated from distance transformation. To make the algorithm valid for different application purposes, both the isosurface and the primitive representations of the object surface are derived. The isosurface is constructed by PDE (Partial Differential Equation), which can be solved iteratively. The traditional distance interpolating method, which was used by several researchers for surface reconstruction, is an approximate solution of the PDE. The primitive representations are approximated by Voronoi Diagram transformation of the surface space. Isosurfaces have the advantage that subsequent geometric analysis of the object can be easily carried out while primitive representation is easy to visualize. The proposed technique allows for surface recovery at any desired resolution, thus avoiding the inherent problems of correspondence, tiling, and branching.

## 1 Introduction

Surface reconstruction from a set of planar sectional contours has been an important problem in diverse scientific fields. These contours define the intersections of the object surface with a set of parallel planes along a desired orientation. For example, CT and MRI techniques can provide dense serial sectional representation of electron density and water molecule concentration at different locations along a particular axis. Similarly, in confocal microscopy, cross sections are obtained by focusing the optical system at specific locations along the $z$ axis. The propose of our work is to utilize these cross sections to recover the three-dimensional surfaces of the object for visualization as well as geometric analysis.

Most of the existing techniques treat the "surface from contours" as a primitive reconstruction problem. The primitives are calculated from the adjacent planar contours according to their geometrical relationship. The approaches lead to three sources of ambiguities [7, 12, 16, 17, 18, 21, 32, 33]: (1) correspondence, (2) tiling, and (3) branching problems. A few techniques aim to represent the surface as the zero-set of an implicit function [15, 20, 25] which can be visualized by, *e.g.,* the matching cubes algorithm. A field function is computed in each slice, and the volume data is constructed by spline interpolation of the slice images. These approaches also lead to some ambiguities: (1) field function, (2) artificial surface and (3) efficiency.

In this paper, we treat the problem in a new way. We derive both the isosurface-based and primitive-based representations of the target object so that the reconstructed surface is efficient for visualization as well as geometric analysis. This approach is based on representing the problem as a partial differential equation (PDE), which can be solved iteratively [10]. The isosurface is calculated by linear interpolation between the distance transformation of adjacent contours while the primitives are computed from the Voronoi Diagram (VD). Although the distance interpolation is used by Jones and Chen [20], it is only an approximation of the solution of the PDE. Our solution naturally avoids the correspondence, tiling, and branching problems in favor of a more robust and efficient solution. The underlying constraint is based on the Equal Importance Criterion (EIC), which suggests that all points contribute equally to the shape-reconstruction process. Formally, the constraint states that surface height decreases linearly along the trajectory of its gradient. As a result, the problem reduces to solving a PDE. Experimental results on both synthetic data and real contours are included.

## 2 Equal Importance Criterion

The proposed reconstruction problem is underconstrained and ill-posed. To constrain the problem, we impose a smoothness measure based on the Equal Importance Criterion (EIC). Consider a pair of contours $\mathcal{C}_1$ and $\mathcal{C}_2$. Let $\mathcal{O}_i(x, y), i = 1, 2$ be the binary "object function" such that $\mathcal{O}_i(x, y) = 1$ if the point $(x, y)$ belongs to the object (inside), $\mathcal{O}_i(x, y) = 0$ if $(x, y)$ is on the curve, and $\mathcal{O}_i(x, y) = -1$, otherwise. The surface space $R(\mathcal{C}_1, \mathcal{C}_2)$ is defined by:

$$R(\mathcal{C}_1, \mathcal{C}_2) = \{(x, y) | \mathcal{O}_1(x, y)\mathcal{O}_2(x, y) \leq 0\} \qquad (1)$$

In $R(\mathcal{C}_1, \mathcal{C}_2)$, we want to construct a surface $f(x, y)$ such that $f(\mathcal{C}_1) = 1, f(\mathcal{C}_2) = 2$. Obviously, in the absence of no constraints, infinitely many solutions exist. To constrain the problem, we assert that *every point in $R(\mathcal{C}_1, \mathcal{C}_2)$ is equally important and contributes similarly to the reconstruction process. Any other assumption means that we know something about the surface.* We call this the *Equal*

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*Importance Criterion*. This constraint is formalized by requiring that the change in the gradient-magnitude along the gradient direction should be zero, that is: $\mathcal{J}_1(f) = \bigtriangledown(| \bigtriangledown f|) \cdot \frac{\bigtriangledown f}{|\bigtriangledown f|} = 0$ where $\bigtriangledown f$ indicates the gradient of $f$, $|\;|$ the norm and $\cdot$ the inner product. The above PDE implies that along each trajectory of the gradient of the surface, the magnitude of the gradient is a constant. In another words, the height decreases linearly from 2 to 1. The level curves of the surface are equally distributed along the gradient direction. Thus, in view of height, which is our only evidence about surface, all points are equally important to us. $\mathcal{J}_1$ can be reduced to the "Infinity Laplacian" $\mathcal{J}_2(f) = f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy} = 0$ which has been studied in the literature [1, 2, 3, 14]. Thus,

$$\begin{aligned} &\mathcal{J}_2(f) = 0, \quad (x, y) \in R(\mathcal{C}_1, \mathcal{C}_2) \\ &\quad s.t. \;\; f(\mathcal{C}_1) = 1, \quad f(\mathcal{C}_2) = 2. \end{aligned} \tag{2}$$

# 3 Isosurface reconstruction

In our PDE-based approach, the correspondence, tiling, and branching problems have been eliminated and the distance between $\mathcal{C}_1$ and $\mathcal{C}_2$ in the $z$ direction is no longer important because it only changes the solution by a scale. We now develop an efficient solution for the above equation.

## 3.1 Solving the PDE

Let's define $\mathcal{D}_i(x, y), i = 1, 2$ as the *Distance Transformation* of curve $\mathcal{C}_i$, where $\mathcal{D}_i(x, y)$ has the same sign of $\mathcal{O}_i(x, y)$. For each point $p$ (shown in Figure 1), there should be a gradient trajectory $\gamma$ passing through it such that it intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ at $p_1$ and $p_2$, respectively. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal height contours, it is easy to show that the normal of these two contours and the gradient of surface are in the same direction. Thus, $\gamma \perp \mathcal{C}_1$ at $p_1$ and $\gamma \perp \mathcal{C}_2$ at $p_2$. We can approximate the curve $\gamma$, passing through $p$, by drawing two line segments $pp'_1 \perp \mathcal{C}_1, pp'_2 \perp \mathcal{C}_2$, to create $p'_1 p p'_2$. Let $l$ denote the length of $\gamma$ from $p_1$ to $p_2$. Hence, $l \approx |p'_1 p| + |p'_2 p|$. The preceding formulation indicates that $|p'_1 p| = -\mathcal{D}_1(p)$, $|p'_2 p| = \mathcal{D}_2(p)$. Since the height decreases linearly, $f$ can be approximated by:

$$f(x, y) = \frac{2|p'_1 p| + |p'_2 p|}{|p'_1 p| + |p'_2 p|} = \frac{\mathcal{D}_2(x, y) - 2\mathcal{D}_1(x, y)}{\mathcal{D}_2(x, y) - \mathcal{D}_1(x, y)} \tag{3}$$

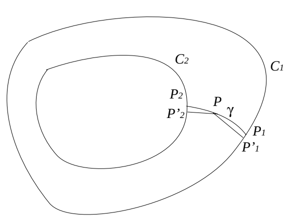


Figure 1: Approximation solution of the PDE.

## 3.2 Isosurface construction

The three-dimensional isosurface representation, $\phi(x, y, z), 1 \leq z \leq 2$, can now be expressed as:

$$\phi(x, y, z) = (z - 1)\mathcal{D}_2(x, y) + (2 - z)\mathcal{D}_1(x, y) \tag{4}$$

This (zero value) isosurface $\phi(x, y, z) = 0$ is:

$$z(x, y) = \frac{\mathcal{D}_2(x, y) - 2\mathcal{D}_1(x, y)}{\mathcal{D}_2(x, y) - \mathcal{D}_1(x, y)} \tag{5}$$

which is exactly the surface that we reconstructed in Equation (3). Note that Equation (4) is exactly the distance interpolation as used in [20]. Equation (4) is better than Equation (3) because it works for any adjacent $\mathcal{C}_1$ and $\mathcal{C}_2$ even if $\mathcal{C}_1 = \mathcal{C}_2$. Thus, our algorithm treats any contour and topological changes naturally and cannot fail. From Equation (5), since $1 \leq z \leq 2$ if and only if $\mathcal{D}_2(x, y)\mathcal{D}_1(x, y) \leq 0$, thus $\phi(x, y, z) = 0$ occurs only in the region $R(\mathcal{C}_1, \mathcal{C}_2)$. Thus, we cannot get an artificial isosurface.

The proposed method can be applied iteratively to every pair of adjacent contours $\mathcal{C}_i, \mathcal{C}_{i+1}, i = 1, ..., m - 1$ for constructing a series of subsurfaces $\mathcal{S}_i$. These subsurfaces, $\mathcal{S}_i : i = 1, ..., m - 1$, form the whole surface, namely $\mathcal{S} = \bigcup_i \mathcal{S}_i$. The final output of the algorithm is a three-dimensional data with new slices inserted between every $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$. If we want a resolution of $\sigma < 1$, say 0.1, along the $z$ direction, reconstruction should include between each pair of adjacent contours $Z - 1$ new slices with $Z = \frac{1}{\sigma}$(an integer $Z$ is expected).

In most cases, the contours are close to one another, thus, the smoothness of the union surface is not a problem. When the contours are considerably apart, the surface may be not smooth at the contour locations. The simplest way to smooth the surface is to convolve $\phi(x, y, z)$ with a small scale three-dimensional Gaussian filter, which is well known as equivalent to move every point on the surface along its normal direction at a speed of its mean curvature [22]. Those points or regions with high curvatures will be smoothed.

# 4 Primitive representation

A three-dimensional triangle is the basic surface patch used in most visualization systems. In this section, we show how to approximate the surface defined by Equation (3) as an assembly of triangles. The basic idea is to partition $R(\mathcal{C}_1, \mathcal{C}_2)$ into two-dimensional triangles and then project them to three-dimensional space.

## 4.1 Partitioning $R(\mathcal{C}_1, \mathcal{C}_2)$

Partitioning is based on the Voronoi Diagram (VD), one of the most fundamental data structures in computational geometry and computer vision [4, 24, 23, 35]. Like most of the previous works [5, 11, 28, 37], we assume that $\mathcal{C}_1, \mathcal{C}_2$ are polygonal curves. The vertices and segments linking them are called elements. The VD is a set of points inside $R(\mathcal{C}_1, \mathcal{C}_2)$, where each of them has at least two closest elements equidistant to it. $R(\mathcal{C}_1, \mathcal{C}_2)$ is divided by its VD into singly-connected regions, called Voronoi Regions (VR), according to the nearest-neighbor-rule. Each point in $R(\mathcal{C}_1, \mathcal{C}_2)$ is associated with the element closest to it, and all the points in one VR have the same closest element. See Figure 2. The "net-like" VD consists of line segments and parabola that are fitted by polygon curves [19, 34, 36]. Thus, each VR becomes a polygon.

An iterative approach then partitions each VR into triangles. The approach begins by randomly finding two non-sequential vertices in one VR so that when linked by a segment, the segment is totally inside the VR. This segment

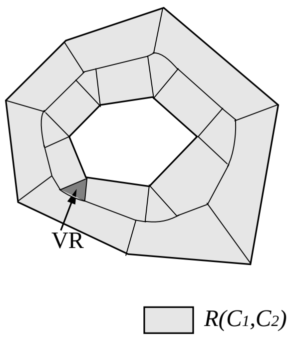



Figure 2: Voronoi diagram.

divides the VR into two parts. The algorithm returns for each part that is a triangle. Otherwise, this part is set as a new polygon, and the process continues recursively.

### 4.2 Reconstructing three-dimensional triangles

The computed two-dimensional triangles are then projected into three-dimensional space, and the corresponding z-values of the vertices (of the triangles) are calculated by Equation (3). An example is shown in Figure 3.

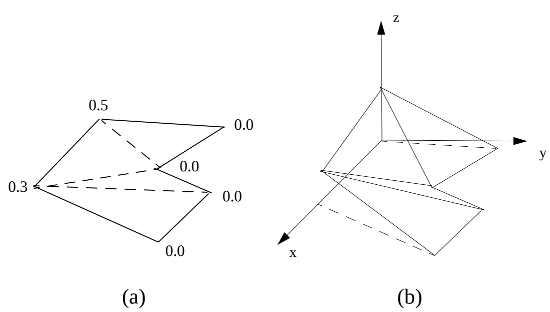



Figure 3: 3D triangle reconstruction. (a) Two-dimensional triangles; and (b) three-dimensional patches.

Two specific situations need more careful treatment. First, if $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at point $p$, then the z-value of $p$ can be either 1 or 2, as shown in Figure 4. In 2D, let $p = (x, y)$. In 3D, $p$ is denoted by $p_1 = (x, y, 1)$ on $\mathcal{C}_1$ and $p_2 = (x, y, 2)$ on $\mathcal{C}_2$. In this case, an additional triangle $b - p_1 - p_2$ besides triangles $a - b - p_2$ and $b - c - p_1$ must be constructed to preserve the continuity of the surface. Second, if $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common segment, as shown in Figure 4, the z-value of that segment can also be either 1 or 2. In 2D, let $a = (x_1, y_1)$, $b = (x_2, y_2)$. In 3D, $a$ and $b$ are denoted by $a_1 = (x_1, y_1, 1), a_2 = (x_1, y_1, 2), b_1 = (x_2, y_2, 1), b_2 = (x_2, y_2, 2)$, respectively. Hence, a rectangle $a_1 - a_2 - b_2 - b_1$ must be constructed.

## 5 Experimental results

The proposed protocol has been tested on real medical images for both the isosurface representation (Figures 5) and the primitive representations (Figure 6).

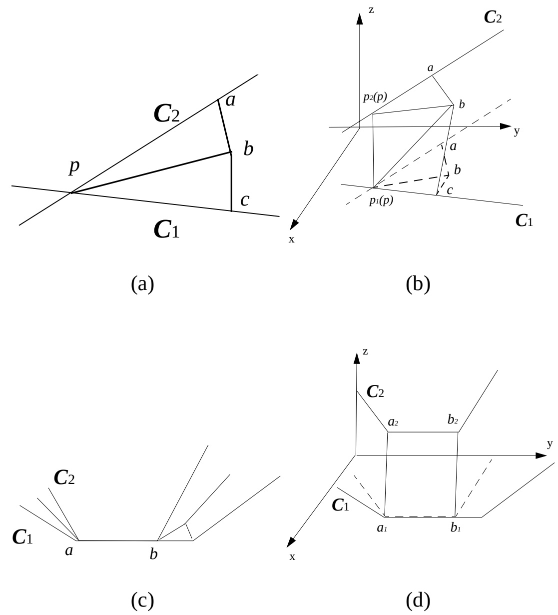



Figure 4: $\mathcal{C}_1, \mathcal{C}_2$ intersect at point $p$. (a)Two-dimensional partition; (b)three-dimensional triangles; (c)Two-dimensional partition; and (d) three-dimensional rectangle.

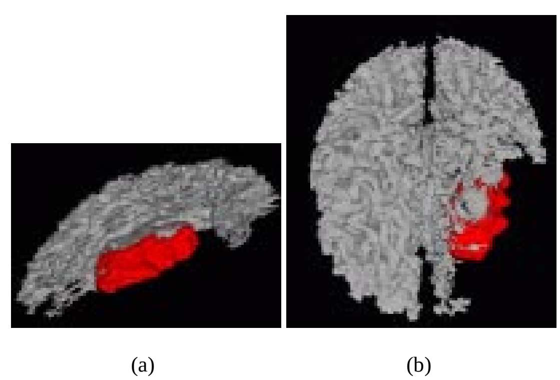
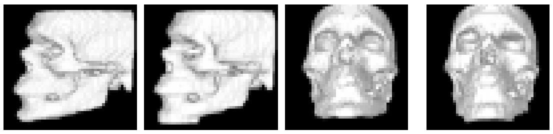



Figure 5: Reconstruction results of white matter in cortex and region due to edema.



(a) (b) (c) (d)

Figure 6: Surface reconstructed from CT data. (a)(c) surface reconstructed from 22 contours; (b)(d) surface reconstructed from 16 contours;

# 6 Conclusion

Shape from cross-sectional contours is an important problem in diverse fields of science and has been studied extensively. However, some of these methods suffer from correspondence, tiling, and branching problems. The novelty of the proposed method is in its unique smoothness measure, the corresponding PDE, and its simple solution based on distance transformation. We showed that a linear solution provides an adequate representation of the isosurface. In the case of primitive representation, VD gives us a natural segmentation of the surface space and enables us to construct small surface patches more easily for any shape. We have tested and verified our approach on data with different degrees of complexities, ranging from simple geometric features to complex and convoluted structures of cortex.

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